Derivation of the Momentum Operator

This set of notes describes one way of deriving the expression for the position-space representation of the momentum operator in quantum mechanics.

First of all, we need to meet a new mathematical friend, the *Dirac delta function* $\delta(x - x_0)$, which is defined by its action when integrated against any function $f(x)$:

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0). \tag{1}$$

In particular, for $f(x) = 1$, we have

$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1. \tag{2}$$

Speaking roughly, one can say that $\delta(x - x_0)$ is a "function" that diverges at the point $x = x_0$ but is zero for all other x, such that the "area" under it is 1.

We can construct some interesting integrals using the Dirac delta function. For example, using Eq. (1), we find

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - y) e^{\frac{-ipx}{\hbar}} dx = \frac{1}{\sqrt{2\pi}} e^{\frac{-ipy}{\hbar}}. \tag{3}$$

Notice that this has the form of a Fourier transform:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{4a}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk, \tag{4b}$$

where $F(k)$ is the Fourier transform of $f(x)$. This means we can rewrite Eq. (3) as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{ip}{\hbar}(x-y)} dx = \delta(x - y). \tag{5}$$

That's enough math; let's start talking physics. The expectation value of the position of a particle, in the position representation (that is, in terms of the position-space wavefunction) is given by

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x) x \psi(x) dx. \tag{6}$$

Similarly, the expectation value of the momentum of a particle, in the momentum representation (that is, in terms of the momentum-space wavefunction) is given by

$$\langle p \rangle = \int_{-\infty}^{\infty} \phi^*(p) p \phi(p) dp. \tag{7}$$

Let's try to calculate the expectation value of the momentum of a particle in the position representation (that is, in terms of $\psi(x)$, not $\phi(p)$).

First of all, we can relate $\psi(x)$ and $\phi(p)$ via a Fourier transform:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(p) e^{\frac{ipx}{\hbar}} dp. \tag{8}$$

Suppose I took a partial derivative of this expression with respect to x. I would get

$$\frac{\partial}{\partial x}\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(p) \left(\frac{\partial}{\partial x} e^{\frac{ipx}{\hbar}} \right) dp = \frac{i}{\hbar\sqrt{2\pi}} \int_{-\infty}^{\infty} p\phi(p) e^{\frac{ipx}{\hbar}} dp, \tag{9}$$

or, with a little rearranging

$$-i\hbar\frac{\partial}{\partial x}\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p\phi(p) e^{\frac{ipx}{\hbar}} dp. \tag{10}$$

We're not done yet! This expression has the form of a Fourier transform, so let's invert it:

$$p\phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -i\hbar\frac{\partial}{\partial x}\psi(x) e^{\frac{-ipx}{\hbar}} dx. \tag{11}$$

This is half of what we need to construct $\langle p \rangle$.

We still need to figure out an expression for the complex conjugate of $\phi(p)$ in terms of x. Again, we turn to the relevant Fourier transform

$$\phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{\frac{-ipx}{\hbar}} dx, \tag{12}$$

and we complex conjugate it to produce

$$\phi^*(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi^*(x) e^{\frac{ipx}{\hbar}} dx. \tag{13}$$

Now, we can put together all these results. The expectation value of the momentum of a particle in the position representation is given by

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \phi^*(p) p \phi(p) dp \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi^*(x) e^{\frac{ipx}{\hbar}} dx \right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -i\hbar \frac{\partial}{\partial y} \psi(y) e^{\frac{-ipy}{\hbar}} dy \right] dp \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x)(-i\hbar) \frac{\partial}{\partial y} \psi(y) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{ip}{\hbar}(x-y)} dp \right] dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x)(-i\hbar) \frac{\partial}{\partial y} \psi(y) \delta(x-y) dx dy \\
&= \int_{-\infty}^{\infty} \psi^*(x) \left[-i\hbar \frac{\partial}{\partial x} \right] \psi(x) dx,
\end{aligned} \tag{14}$$

where we were obligated to change the dummy variable of integration from x to y in the second bracketed expression in the first step to keep the two integrals distinct from each other. We've done it! Furthermore, we have found an expression of the momentum operator in the position representation that works in any quantum mechanics equation that involves p:

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}. \tag{15}$$